

# Bell inequality for qubits based on the Cauchy-Schwarz inequality

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We develop a systematic approach to establish Bell inequalities for qubits based on the Cauchy-Schwarz inequality. We also use the concept of *distinct “roots”* of Bell function to classify some well-known Bell inequalities for qubits. As applications of the approach, we present three new and tight Bell inequalities for four and three qubits, respectively.

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Bell inequality has been regarded as “the most profound discovery in science” [1]. It is at the heart of the study of nonlocality and is the most famous legacy of the late physicist John S. Bell [2]. The inequality shows that the predictions of quantum mechanics are not intuitive, and touches upon fundamental philosophical issues that relate to modern physics. Bell-test experiments serve to investigate the validity of the entanglement effect in quantum mechanics by using some kinds of Bell inequalities, however they to date overwhelmingly show that Bell inequalities are violated. These experimental results provide empirical evidence against local realism [3] and in favor of quantum mechanics.

Bell’s applaudable progress has stirred a great furor. Many people have been attracted in this problem and extensive work on Bell inequalities has been done, including both theoretical analysis and experimental test. The famous Clauser-Horne-Shimony-Holt (CHSH) [4] inequality is a kind of improved Bell inequality that is more convenient for experiments. Now, it is well-known that all pure entangled states of two two-dimensional systems (i.e., qubits) violate the CHSH inequality and the maximum quantum violation is the so-called Tsirelson’s bound  $2\sqrt{2}$ . Mermin, Ardehali, Belinskii, and Klyshko have separately generalized the CHSH inequality to the  $N$ -qubit case, which now known as the MABK inequality, and proved that quantum violation of this inequality increases with the number of particles [5]. In 2001, Scarani and Gisin [6] noticed that the generalized Greenberger-Horne-Zeilinger (GHZ) states [7]:  $|\psi\rangle_{GHZ} = \cos\xi|0\cdots 0\rangle + \sin\xi|1\cdots 1\rangle$  do not violate the MABK inequality for  $\sin 2\xi \leq 1/\sqrt{2^{N-1}}$  (The GHZ state is for  $\xi = \pi/4$ ). In Refs. [8, 9] a general correlation-function  $N$ -qubit Bell inequality has been derived, hereafter we call it as the Werner-Wolf-Żukowski-Brukner (WWZB) inequality. The WWZB inequality includes the MABK inequality as a special case. Ref. [10] shows that (a) For  $N = \text{even}$ , although the generalized GHZ state does not violate the MABK inequality, it does violate the WWZB inequality, and (b) For  $\sin 2\xi \leq 1/\sqrt{2^{N-1}}$  and

$N = \text{odd}$ , the WWZB inequality cannot yet be violated for the whole region  $\xi \in [0, \pi/2]$ . For the three-qubit case, such a difficulty has been overcome in Ref. [11], where a probabilistic Bell inequality was proposed and consequently Gisin’s theorem for three qubits naturally returned. Recent development also indicates that Bell inequality is not unique when one studies Gisin’s theorem for three qubits, in Ref. [12] three of such inequalities have been listed and compared. All the inequalities mentioned above belong to the two-setting Bell inequalities for  $N$  qubits, i.e., they are based on the standard Bell experiment, in which each local observer is given a choice between two dichotomic observables. Some significant generalizations have been made for multi-setting Bell inequalities for  $N$  qubits [13] as well as two-setting Bell inequalities for high-dimensional systems [14, 15]. Notably, in Ref. [16] a multi-setting Bell inequality that violate the generalized GHZ state for the whole region  $\xi \in [0, \pi/2]$  was found.

In this paper, we shall focus on Bell inequality for qubits. It is natural to ask two questions: (i) As an inequality, does Bell inequality have any deep connections with some ancient mathematic inequalities, such as the Cauchy-Schwarz inequality or more generally, the Hölder inequality? (ii) So far, many kinds of Bell inequalities for qubits have emerged, even for three qubits. Can these inequalities be classified in an efficient way? The purpose of this paper is to (i) develop a systematic approach to establish Bell inequalities for qubits based on the Cauchy-Schwarz inequality, and (ii) classify some well-known Bell inequalities based on their distinct “roots”. Let us start from the weighed Hölder inequality since it is a more general inequality which contains the Cauchy-Schwarz inequality as a special case.

*The Cauchy-Schwarz inequality and Bell inequalities.* Let  $f(\lambda)$  and  $g(\lambda)$  be any two real functions for which  $|f(\lambda)|^p$  and  $|g(\lambda)|^q$  are integrable in  $\Gamma$  with  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then the weighed Hölder inequality reads [17]:

$$\int_{\Gamma} fg\rho(\lambda)d\lambda \leq \left[ \int_{\Gamma} |f|^p \rho(\lambda)d\lambda \right]^{\frac{1}{p}} \left[ \int_{\Gamma} |g|^q \rho(\lambda)d\lambda \right]^{\frac{1}{q}}, \quad (1)$$

where  $\Gamma$  is the total  $\lambda$  space and  $\rho(\lambda)$  is a statistical distribution of  $\lambda$ , which satisfies  $\rho(\lambda) \geq 0$  and  $\int_{\Gamma} d\lambda \rho(\lambda) =$

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1. When  $p = q = 2$ , the above inequality reduces to the Cauchy-Schwarz inequality:  $[\int_{\Gamma} fg\rho(\lambda)d\lambda]^2 \leq [\int_{\Gamma} |f|^2\rho(\lambda)d\lambda][\int_{\Gamma} |g|^2\rho(\lambda)d\lambda]$ . It may be very interesting to consider the cases when  $p, q$  take various values. However, in this paper, we restrict our study to the case with  $p = q = 2$ , which is easier for the calculations.

Consider  $N$  spatially separated parties and allow each of them to choose independently among  $M$  observables, determined by some local parameters denoted by  $\lambda$ . Let  $X_j(\mathbf{n}_{k_j}, \lambda)$ , or  $X_{j,k_j}$  for simplicity, denote observables on the  $j$ -th qubit, each of which has two possible outcomes  $-1$  and  $1$ . From the viewpoint of local realism, the values of  $X_j$ 's are predetermined by the local hidden variable (LHV)  $\lambda$  before measurement, and independent of any measurements, orientations or actions performed on the other parties at spacelike separation. The correlation function, in the case of a local realistic theory, is then defined as  $Q(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}, \dots, \mathbf{n}_{k_j}) = \int_{\Gamma} \prod_{j=1}^N X_j(\mathbf{n}_{k_j}, \lambda)\rho(\lambda)d\lambda$ , where  $j = 1, 2, \dots, N$  and  $k_j = 1, 2, \dots, M$ . For convenience, we denote the correlation function  $Q(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}, \dots, \mathbf{n}_{k_j})$  as  $Q_{k_1 k_2 \dots k_j}$ . In the following, we present a systematic approach to establish Bell inequalities for qubits based on the Cauchy-Schwarz inequality. It contains three main steps:

*Step 1: Connecting functions  $f(\lambda)$  and  $g(\lambda)$  with the observables.*  $f(\lambda)$  and  $g(\lambda)$  in inequality (1) can be functions of the observables of the parties. To express the functions more concisely, let  $X_{j,0} = 1$ , then one has

$$f(\lambda) = \sum_{\chi} C_{\chi} \prod_{j=1}^N X_{j,k_j}, \quad g(\lambda) = \sum_{\chi} D_{\chi} \prod_{j=1}^N X_{j,k_j}. \quad (2)$$

Here we associate to each observables,  $X_{1,1}$ ,  $X_{1,1}X_{2,1}$ , or generally  $\prod_{j=1}^N X_{j,k_j}$ , a single symbol  $\chi$ , which stands for  $N$  pairs of indices (one pair for each observer). Obviously, there are  $N_{\chi} = (1 + M)^N$  distinct values of  $\chi$ . The constant numbers  $C_{\chi}$  and  $D_{\chi}$  are coefficients of  $\prod_{j=1}^N X_{j,k_j}$ .

*Step 2: Establishing Bell inequalities by determining coefficients  $C_{\chi}$  and  $D_{\chi}$ .* Note that for the qubit case, one always has  $X_{j,k_j}^2 = 1$ , which is useful for simplifying  $f(\lambda)g(\lambda)$ ,  $f^2(\lambda)$  and  $g^2(\lambda)$  in the inequality (1). One will find that some terms in  $f(\lambda)g(\lambda)$ ,  $f^2(\lambda)$  and  $g^2(\lambda)$ , such as  $X_{1,1}X_{1,2}$ ,  $X_{1,1}X_{1,2}X_{2,1}$ , or  $X_{1,1}X_{1,2}X_{2,1}X_{2,2}$ , etc., are impossible to calculate in quantum mechanics. Consequently, we shall set all the coefficients of such terms be zero, and then get a series of equations for  $C_{\chi}$  and  $D_{\chi}$ . By solving these equations, we obtain the solutions of  $C_{\chi}$  and  $D_{\chi}$ . Substituting these solutions into Eqs. (1) (2), one gets a set of Bell inequalities:  $\langle f(\lambda)g(\lambda) \rangle_{LHV}^2 \leq \langle f(\lambda) \rangle_{LHV}^2 \langle g(\lambda) \rangle_{LHV}^2$ . Here  $\langle f(\lambda)g(\lambda) \rangle_{LHV} = \int_{\Gamma} f(\lambda)g(\lambda)\rho(\lambda)d\lambda$ , and  $\langle f(\lambda) \rangle_{LHV}^2$ ,  $\langle g(\lambda) \rangle_{LHV}^2$  are defined similarly.

*Step 3: Ruling out the trivial Bell inequalities.* Some inequalities obtained in *Step 2* are trivial, i.e., they cannot be violated in quantum mechanics. Thus we should rule them out by calculating the quantum violation of

each inequality. In this step, we finally achieve some nontrivial Bell inequalities, such as the tight inequalities.

Here we present an example to illuminate this method.

*Example 1: Derivation of the CHSH inequality.* Let us look at the simplest case, i.e.,  $N = 2, M = 2$ . Let

$$\begin{aligned} f(\lambda) &= C_0 + C_1[X_{1,1} + X_{2,1}] + C_2[X_{1,2} + X_{2,2}] \\ &\quad + C_3X_{1,1}X_{2,1} + C_4[X_{1,1}X_{2,2} + X_{1,2}X_{2,1}] \\ &\quad + C_5X_{1,2}X_{2,2}, \\ g(\lambda) &= D_0 + D_1[X_{1,1} + X_{2,1}] + D_2[X_{1,2} + X_{2,2}] \\ &\quad + D_3X_{1,1}X_{2,1} + D_4[X_{1,1}X_{2,2} + X_{1,2}X_{2,1}] \\ &\quad + D_5X_{1,2}X_{2,2}, \end{aligned} \quad (3)$$

then we get a series of equations of  $C_j$ 's and  $D_j$ 's (here we omit them for sententiousness). After solving these equations, we finally choose the solutions:  $C_0$  and  $D_0$  are arbitrary real numbers,  $C_1 = C_2 = 0, C_3 = C_4 = -C_5, D_1 = D_2 = 0, D_3 = D_4 = -D_5$ , then from  $\langle f(\lambda)g(\lambda) \rangle^2 \leq \langle f(\lambda) \rangle^2 \langle g(\lambda) \rangle^2$  we have a Bell inequality as:  $(Q_{11} + Q_{12} + Q_{21} - Q_{22})^2 \leq 4$ , or  $(Q_{11} + Q_{12} + Q_{21} - Q_{22})/2 \leq 1$ . It is nothing but the famous CHSH inequality!

Similarly, the MABK and the WWZB inequalities can also be derived with the same approach, although the computation becomes a bit more complicated when  $N$  increases. However, the above approach can be improved further with the aid of the distinct “roots” of the Bell function. What do we mean the “roots” of the Bell function? For instance, let the left-hand side of the CHSH inequality  $\mathcal{B}(\lambda) = (X_{1,1}X_{2,1} + X_{1,1}X_{2,2} + X_{1,2}X_{2,1} - X_{1,2}X_{2,2})/2$  be the Bell function, for each set of values [such as  $\{X_{1,1} = 1, X_{1,2} = 1, X_{2,1} = 1, X_{2,2} = 1\}$ ], the Bell function corresponds to a number [such as  $\mathcal{B}(\lambda) = 1$ ]. This number is called a “root” of the Bell function  $\mathcal{B}(\lambda)$ . Obviously, there are totally  $2^4 = 16$  sets of values  $\{X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}\}$ , so  $\mathcal{B}(\lambda)$  has totally 16 roots. However, 8 roots equal to  $-1$ , the other 8 roots equal to  $1$ , therefore  $\mathcal{B}(\lambda)$  has only two distinct “roots”:  $\Lambda_1 = -1$  and  $\Lambda_2 = 1$ . Then for any set of values  $\{X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}\}$ , one always has the algebraic equation:  $[\mathcal{B}(\lambda) - \Lambda_1][\mathcal{B}(\lambda) - \Lambda_2] = 0$ , or  $\mathcal{B}^2(\lambda) = 1$ . We have the following Theorem.

*Theorem 1.* Let  $S_2 = \{\mathcal{B}(\lambda) \mid \mathcal{B}^2(\lambda) = 1\}$ , for  $\forall \mathcal{B}(\lambda) \in S_2$ , we have Bell inequality  $\mathcal{I} = |\langle \mathcal{B}(\lambda) \rangle_{LHV}| \leq 1$ .

*Proof.* Let  $f(\lambda) = 1 + \mathcal{B}(\lambda)$ ,  $g(\lambda) = 1 - \mathcal{B}(\lambda)$ , then from the Cauchy-Schwarz inequality we have  $[\int [1 + \mathcal{B}(\lambda)][1 - \mathcal{B}(\lambda)]\rho(\lambda)d\lambda] \leq \int [1 + \mathcal{B}(\lambda)]^2\rho(\lambda)d\lambda \int [1 - \mathcal{B}(\lambda)]^2\rho(\lambda)d\lambda$ , which yields  $\langle \mathcal{B}(\lambda) \rangle_{LHV}^2 \leq \langle \mathcal{B}^2(\lambda) \rangle_{LHV}$ . Because  $\mathcal{B}(\lambda) \in S_2$ , which implies  $\mathcal{B}^2(\lambda) = 1$ , thus we arrive at the Bell inequality  $|\langle \mathcal{B}(\lambda) \rangle_{LHV}| \leq 1$ . This ends the proof.

Based on *Theorem 1*, there is a simpler way to derive Bell inequality as follows: Let

$$\mathcal{B}(\lambda) = \frac{1}{2} \left[ X(\lambda) + Y(\lambda) + Z(\lambda) - X(\lambda)Y(\lambda)Z(\lambda) \right], \quad (4)$$

one easily proves that  $\mathcal{B}^2(\lambda) = 1$ , provided  $X^2(\lambda) = Y^2(\lambda) = Z^2(\lambda) = 1$ , then from *Theorem 1* one has

a Bell inequality  $\mathcal{I} = |\langle \mathcal{B}(\lambda) \rangle_{LHV}| \leq 1$ . For instance, let  $X(\lambda) = A_1(\lambda)B_1(\lambda)$ ,  $Y(\lambda) = A_1(\lambda)B_2(\lambda)$ ,  $Z(\lambda) = A_2(\lambda)B_1(\lambda)$ , which yields  $X(\lambda)Y(\lambda)Z(\lambda) = A_2(\lambda)B_2(\lambda)$  [using  $A_j^2(\lambda) = 1$ ,  $B_j^2(\lambda) = 1$ ], then one has the CHSH inequality:  $\mathcal{I}_{CHSH} = |\langle \mathcal{B}(\lambda) \rangle_{LHV}| = |\langle A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \rangle/2| \leq 1$ ; Let  $X(\lambda) = A_1(\lambda)B_1(\lambda)C_2(\lambda)$ ,  $Y(\lambda) = A_1(\lambda)B_2(\lambda)C_1(\lambda)$ ,  $Z(\lambda) = A_2(\lambda)B_1(\lambda)C_1(\lambda)$ , one has the MABK inequality for three qubits as  $\mathcal{I}_{MABK} = |\langle \mathcal{B}(\lambda) \rangle_{LHV}| = |\langle A_1B_1C_2 + A_1B_2C_1 + A_2B_1C_1 - A_2B_2C_2 \rangle/2| \leq 1$ .

The Bell functions of the MABK and the WWZB inequalities for  $N$  qubits have two distinct “roots” 1 and  $-1$ , so they belong to  $S_2$ . The Bell function  $\mathcal{B}(\lambda)$  in Eq. (4) naturally connects with the MABK inequality or the WWZB inequality in the following way: Let  $X(\lambda) = \mathcal{B}_{N-1}(\lambda)X_{N,1}$ ,  $Y(\lambda) = \mathcal{B}_{N-1}(\lambda)X_{N,2}$ ,  $Z(\lambda) = \mathcal{B}'_{N-1}(\lambda)X_{N,1}$ , where  $\mathcal{B}_{N-1}(\lambda)$  is the Bell function of MABK inequality or the WWZB inequality for  $(N-1)$  qubits, and  $\mathcal{B}'_{N-1}(\lambda)$  is obtained through the interchanges  $X_{j,1} \leftrightarrow X_{j,2}$ . After substituting them into Eq. (4) and using  $X(\lambda)Y(\lambda)Z(\lambda) = \mathcal{B}'_{N-1}(\lambda)X_{N,2}$ , one has  $\mathcal{B}(\lambda) = \frac{1}{2}\{\mathcal{B}_{N-1}(\lambda)[X_{N,1} + X_{N,2}] + \mathcal{B}'_{N-1}(\lambda)[X_{N,1} - X_{N,2}]\}$ , which is just the Bell function of MABK and WWZB inequalities. By the way, if  $\mathcal{B}'_{N-1}(\lambda)$  is replaced by identity  $\mathbf{1}_{N-1} = \prod_{j=1}^{N-1} X_{j,0}$ , then one recovers the family of two-setting Bell inequality for many qubits as shown in Ref. [18] (see inequality (4) in [18]). Also, both the three-setting Bell inequality (28) and the four-setting Bell inequality (35) presented in Ref. [19] belong to  $S_2$ .

For Bell functions with three or more distinct “roots”, we have the following Theorem.

*Theorem 2.* Let  $S_3 = \{\mathcal{B}(\lambda) \mid \mathcal{B}^3(\lambda) = \mathcal{B}(\lambda)$ ,  $\mathcal{B}(\lambda) \notin S_2\}$ , i.e.,  $\mathcal{B}(\lambda)$  must have three distinct “roots”  $\Lambda_1 = -1$ ,  $\Lambda_2 = 0$ , and  $\Lambda_3 = 1$ , then for  $\forall \mathcal{B}(\lambda) \in S_3$ , one has the Bell inequality:  $|\langle \mathcal{B}(\lambda) \rangle_{LHV}| \leq 1$ . In general, if  $S_n = \{\mathcal{B}(\lambda) \mid \prod_{j=0}^{n-1} (\mathcal{B} - \Lambda_j) = 0$ ,  $\mathcal{B}(\lambda) \notin \bigcup_{k=2}^{n-1} S_k$ ,  $n \in \text{integers}$ ,  $n \geq 3\}$ , which means that  $n$  “roots” of  $\mathcal{B}(\lambda)$  uniformly distribute between  $-1$  and  $1$  with  $\Lambda_j = -1 + 2j/(n-1)$ , for  $\forall \mathcal{B}(\lambda) \in S_n$ , one has the Bell inequality:  $|\langle \mathcal{B}(\lambda) \rangle_{LHV}| \leq 1$ .

*Proof.* First, we prove that for  $n = 3$  the theorem is valid. Since  $\langle \mathcal{B}(\lambda) \rangle^2 \leq \langle \mathcal{B}^2(\lambda) \rangle$ , what we need to do is to prove  $\langle \mathcal{B}^2(\lambda) \rangle \leq 1$ . Let  $f(\lambda) = 1 + \mathcal{B}^2(\lambda)$ ,  $g(\lambda) = 1 - \mathcal{B}^2(\lambda)$ , from the Cauchy-Schwarz inequality one obtains  $\langle \mathcal{B}^2(\lambda) \rangle^2 \leq \langle \mathcal{B}^4(\lambda) \rangle$ . By using  $\mathcal{B}(\lambda)^3 = \mathcal{B}(\lambda)$ , we have  $\langle \mathcal{B}^2(\lambda) \rangle^2 \leq \langle \mathcal{B}^4(\lambda) \rangle = \langle \mathcal{B}^2(\lambda) \rangle$ , i.e.,  $\langle \mathcal{B}^2(\lambda) \rangle[\langle \mathcal{B}^2(\lambda) \rangle - 1] \leq 1$ . Because  $\langle \mathcal{B}^2(\lambda) \rangle \geq 0$ , then we have  $\langle \mathcal{B}^2(\lambda) \rangle_{LHV} \leq 1$ . Second, we use the induction method to prove this theorem. Suppose for all  $n \leq k$ , the theorem is valid, then for  $\mathcal{B}_{k+1} \in S_{k+1}$ , one has  $\mathcal{B}_{k+1} = (\mathcal{B}_{k+1}^2 - 1) \prod_{j=1}^{k-1} (\mathcal{B}_{k+1} - \Lambda_j) = 0$ . If  $(\mathcal{B}_{k+1}^2 - 1) = 0$ , Theorem 1 yields  $|\langle \mathcal{B}_{k+1} \rangle| \leq 1$ . If  $\prod_{j=1}^{k-1} (\mathcal{B}_{k+1} - \Lambda_j) = 0$ , from the principle of induction method, one obtains  $|\langle \mathcal{B}_{k+1} \rangle| = |\langle \frac{k-2}{k} \mathcal{B}_{k-1} \rangle| \leq \frac{k-2}{k}$ . Thus, one has  $|\langle \mathcal{B}(\lambda) \rangle_{LHV}| \leq 1$ , which completes the proof.

Let us see what Bell inequalities appeared in litera-

ture belong to  $S_3$ . The first example is the three-setting Bell inequality for  $N$  qubits proposed in Ref. [20] [see inequality (9) in [20] or inequality (2) in [21]]. Although this inequality is not tight, it has a particular property that for the GHZ state it is more resistant to noise than the MABK inequality when  $N \geq 4$ . The second example is the two two-setting Bell inequalities listed in Ref. [12] [see inequalities (3) and (4) in [12]]; These two inequalities are not tight but they are violated by any pure entangled state of three qubits. Moreover, there indeed exist some Bell inequalities belonging to  $S_4$ . The first example is a three-setting Bell inequality for two qubits proposed in Ref. [22] [see inequality (19) in [22]]. It is a relevant two-qubit Bell inequality inequivalent to the CHSH inequality. The most interesting feature of this tight inequality is that there exist states that violate it, but do not violate the CHSH inequality. The second example is the two-setting Bell inequality presented in Ref. [12] [see inequality (8) in [12]]. It is a tight Bell inequality and is violated by any pure entangled state of three qubits. To our knowledge, Bell inequalities belonging to  $S_n$  ( $n \geq 5$ ) have seldom appeared in the literature.

As applications of the approach, we present three new and tight Bell inequalities that belong to  $S_3$  as follows.

*Example 2: A new two-setting Bell inequality for four qubits*  $\langle I_{4,2} \rangle \leq 1$ . Let  $\mathcal{B}(\lambda) = \frac{7}{16} + \frac{9}{16}I_{4,2}(\lambda)$ , where  $I_{4,2} = (1/9)(-5Q_{1111} - 2Q_{2222} - [Q_{1120} + Q_{1210} + Q_{2110} + Q_{1102}] - [Q_{2200} + Q_{2020} + Q_{2002} + Q_{0220} + Q_{0202} + Q_{0022}] + [Q_{2000} + Q_{0200} + Q_{0020} + Q_{0002}] + 2[Q_{1000} + Q_{0100} + Q_{0010} + Q_{0001}] + [Q_{1110} + Q_{1101} + Q_{1011} + Q_{0111}] + [Q_{1201} + Q_{2101} + Q_{1012} + Q_{1102} + Q_{2011} + Q_{0112} + Q_{0121} + Q_{0211}] + 2[Q_{1112} + Q_{1121} + Q_{1211} + Q_{2111}] - [Q_{1220} + Q_{2120} + Q_{2210} + Q_{1202} + Q_{2102} + Q_{2201} + Q_{1022} + Q_{2021} + Q_{0122} + Q_{0221} + Q_{0212}] + [Q_{1122} + Q_{1212} + Q_{1221} + Q_{2211} + Q_{2121} + Q_{2112}])$ ,  $Q_{mnkl} = \int_{\Gamma} X_{1,m}X_{2,n}X_{3,k}X_{4,l}\rho(\lambda)d\lambda$ , ( $m, n, k, l = 0, 1, 2$ ), are the correlation functions, and here we have set  $X_{j,0} = 1$  ( $j = 1, 2, 3, 4$ ). It is easy to check  $\mathcal{B}(\lambda) \in S_3$ , therefore from Theorem 2 we have the Bell inequality  $|\langle \mathcal{B}(\lambda) \rangle_{LHV}| \leq 1$ , or  $\langle I_{4,2} \rangle \leq 1$ . This inequality  $\langle I_{4,2} \rangle \leq 1$  is tight and is symmetric under the permutations of  $X_{j,k}$ 's.

For four qubits, the Werner states read  $\rho_W = V|\psi\rangle\langle\psi| + (1-V)\rho_{noise}$ , where  $|\psi\rangle = (1/\sqrt{2})(|0000\rangle + |1111\rangle)$  is the four-qubit GHZ state,  $V$  is the so-called visibility,  $\rho_{noise} = \mathbb{I}_{16}/16$ , and  $\mathbb{I}_{16}$  is the  $16 \times 16$  identity matrix. In quantum mechanics, the observables read  $X_{j,k} = \vec{\sigma} \cdot \vec{n}_{k,j}$ , ( $j = 1, 2, 3, 4$ ;  $k = 1, 2$ ),  $\vec{\sigma}$  is the vector of Pauli matrices,  $\vec{n}_{k,j} = (\sin \theta_{k,j} \cos \phi_{k,j}, \sin \theta_{k,j} \sin \phi_{k,j}, \cos \theta_{k,j})$ ,  $X_{j,0} = \mathbb{I}_2$  is the  $2 \times 2$  identity matrix. The correlation functions for the Werner states read  $Q_{mnkl} = \text{Tr}(\rho_W X_{1,m} \otimes X_{2,n} \otimes X_{3,k} \otimes X_{4,l})$ . For the GHZ state, the threshold visibility is  $V_{GHZ}^{4\text{-qubit}} \approx 9/15.56 \approx 0.5784$ , which is larger than  $1/2\sqrt{2}$ , the result given by the Mermin inequality. Namely, the inequality  $\langle I_{4,2} \rangle \leq 1$  is less resistant to noise. However, numerical result shows that the generalized GHZ states violate  $\langle I_{4,2} \rangle \leq 1$  for the whole region, which cannot be done for the MABK inequality. By setting  $X_{4,1} = X_{4,2} = 1$ ,  $\langle I_{4,2} \rangle \leq 1$  reduces to an equivalent form of the three-qubit Bell inequality

(4) in Ref. [12]. Remarkably, numerical calculation indicates that Bell inequality  $\langle I_{4,2} \rangle \leq 1$  is violated by all pure entangled states of four qubits, thus it is a good candidate for proving Gisin's theorem for four qubits.

*Example 3: Another new two-setting Bell inequality for four qubits*  $\langle I'_{4,2} \rangle \leq 1$ . Let  $\mathcal{B}(\lambda) = \frac{6}{16} + \frac{10}{16}I'_{4,2}(\lambda)$ , where  $I'_{4,2} = (1/10)(-[Q_{1200} + Q_{2100} + Q_{1020} + Q_{2010} + Q_{1002} + Q_{2001} + Q_{0120} + Q_{0210} + Q_{0102} + Q_{0201} + Q_{0012} + Q_{0021}] - Q_{2222} + [Q_{1112} + Q_{1121} + Q_{1211} + Q_{2111}] + 3[Q_{1000} + Q_{0100} + Q_{0010} + Q_{0001}] + [Q_{2000} + Q_{0200} + Q_{0020} + Q_{0002}] - [Q_{1220} + Q_{2120} + Q_{2210} + Q_{1202} + Q_{2102} + Q_{2201} + Q_{1022} + Q_{2021} + Q_{2012} + Q_{0122} + Q_{0221} + Q_{0212}] + [Q_{1122} + Q_{1212} + Q_{1221} + Q_{2211} + Q_{2121} + Q_{2112}] - [Q_{2220} + Q_{2202} + Q_{2022} + Q_{0222}] - 3Q_{1111})$ . One may verify that  $\mathcal{B}(\lambda) \in S_3$ , thus we have the Bell inequality  $\langle I'_{4,2} \rangle \leq 1$ . This inequality is tight and is violated by the generalized GHZ states for the whole region. It is also a good candidate for proving the Gisin's theorem for four qubits.

*Example 4: A new three-setting Bell inequality for three qubits*  $\langle I_{3,3} \rangle \leq 1$ . The Bell inequality reads

$$\begin{aligned} \langle I_{3,3} \rangle = & [(Q_{223} + Q_{232} + Q_{322}) - 2(Q_{211} + Q_{121} + Q_{112}) \\ & + (Q_{221} + Q_{122} + Q_{212}) - (Q_{331} + Q_{313} + Q_{133}) \\ & + (Q_{321} + Q_{312} + Q_{213} + Q_{231} + Q_{123} + Q_{132}) \\ & + 2Q_{111} + 4Q_{222} - Q_{333}]/8 \leq 1, \end{aligned} \quad (5)$$

and  $\mathcal{B}(\lambda) = I_{3,3}(\lambda) \in S_3$ . This inequality is also tight, which reduces to the CHSH inequality when  $X_{3,1} = 1$ ,  $X_{3,2} = X_{3,3} = -1$ . It is resistant to noise as strong as the MABK inequality for the GHZ state (i.e.,  $\langle I_{3,3}^{max} \rangle = 2$ ), but is violated by the generalized GHZ states for a wider region (for instance,  $\langle I_{3,3}^{max} \rangle \approx 1.0059$  when  $\xi = \pi/12$ ).

In conclusion, we have developed a systematic ap-

proach to establish Bell inequalities for qubits based on the Cauchy-Schwarz inequality. We have also used the concept of *distinct "roots"* of Bell function to classify some well-known Bell inequalities for qubits. As applications of the approach, we have presented three new and tight Bell inequalities. In addition, there is an alternative way to derive the CHSH inequality: From *Theorem 1* and Eq. (4), one may have  $\mathcal{I}_1 = \langle 1 + A_1B_1 - A_2B_2 + A_1A_2B_1B_2 \rangle / 2 \leq 1$ ,  $\mathcal{I}_2 = \langle 1 + A_1B_2 + A_2B_1 - A_1A_2B_1B_2 \rangle / 2 \leq 1$ . By adding up these two inequalities, one arrives at the famous CHSH inequality:  $\mathcal{I}_{CHSH} = \mathcal{I}_1 + \mathcal{I}_2 = \langle A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \rangle / 2 \leq 1$ . Usually, combining two inequalities directly will lead to an inequality with looser constraint than before. In such a sense, inequalities  $\mathcal{I}_1$  and  $\mathcal{I}_2$  are stronger than the CHSH inequality, but the difficult point is that it is impossible to compute the term  $\langle A_1A_2B_1B_2 \rangle$  for quantum mechanics. Some time ago, Gisin posed a question to find Bell inequalities which are more efficient than the CHSH one for Werner states [23] (see also [24]). Recently, Vértesi has given a positive answer to Gisin's question by providing a new family of multi-setting (at least  $M = 465$  settings for each party) Bell inequalities [25], which proves the two-qubit Werner states to be nonlocal for a wider parameter range  $0.7056 < V \leq 1$ . How to use *Theorems 1* and *2* to construct even more efficient Bell inequalities for Werner states is a significant topic, which we shall investigate subsequently.

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